

SIMPLE TRANSITIVE 2-REPRESENTATIONS VIA (CO)ALGEBRA 1-MORPHISMS

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ABSTRACT. For any fiat 2-category \mathcal{C} , we show how its simple transitive 2-representations can be constructed using coalgebra 1-morphisms in the injective abelianization of \mathcal{C} . Dually, we show that these can also be constructed using algebra 1-morphisms in the projective abelianization of \mathcal{C} . We also extend Morita–Takeuchi theory to our setup and work out several examples, including that of Soergel bimodules for dihedral groups, explicitly.

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1. INTRODUCTION

The subject of 2-representation theory, which has its origins in [CR, KhLa, Ro], is the higher categorical analogue of the classical representation theory of algebras. A systematic study of the “finite dimensional” counterpart of 2-representation theory started in [MM1], and was continued in [MM2]–[MM6], see also [Xa, Zh1, Zh2, MZ] and the references therein. In particular, for a given finitary 2-category \mathcal{C} , the paper [MM5] defines an appropriate 2-analog of simple representations, the so-called *simple transitive* 2-representations, and uses these 2-representations to establish a Jordan–Hölder theory for finitary 2-categories. This motivates the problem of classifying the simple transitive 2-representations of a given finitary 2-category \mathcal{C} . For instance, this question was studied, for various 2-categories, in e.g. [MM5, MM6, Zh2, Zi, MZ, MaMa, KMMZ, MT].

An important example of such simple transitive 2-representations is provided by the so-called *cell 2-representations* defined in [MM1, MM2] using combinatorics of 1-morphisms in \mathcal{C} . The notion of cell 2-representation is inspired by the Kazhdan–Lusztig cell representations of Hecke algebras of Coxeter groups [KL]. In some cases, for example for the 2-categories of Soergel bimodules in type A , cell 2-representations exhaust simple transitive 2-representations, as was shown in [MM5]. However, it turns out that, in many cases, there are simple transitive 2-representations which are not equivalent to cell 2-representations. The first, very degenerate, examples already appeared in [MM5]. However, the first interesting, and unexpected, example appeared in [MaMa] which studies simple transitive 2-representations for some subquotients of Soergel bimodules in dihedral Coxeter types $I_2(4)$ and $I_2(5)$.

The subsequent paper [KMMZ] studies the classification of simple transitive 2-representations for so-called *small quotients of Soergel bimodules* in all finite Weyl types. In particular, the existence of simple transitive 2-representations which are not cell 2-representations was established for all dihedral Coxeter types $I_2(2n)$, where $n > 2$. The classification problem was completed for all types with the exception of $I_2(12)$, $I_2(18)$ and $I_2(30)$.

The classification of simple transitive 2-representations is usually approached in two steps. The first step addresses the classification of certain integral representations of the group algebra of the corresponding Weyl group. As it turns out in [KMMZ], for dihedral types, this latter classification is given in terms of simply laced Dynkin diagrams. The type A Dynkin diagrams lead to cell 2-representations, and the type D Dynkin diagrams lead to new simple transitive 2-representations which, together with the cell 2-representations, exhaust all simple transitive 2-representations unless

the dihedral group is of type $I_2(12)$, $I_2(18)$ or $I_2(30)$. (These three cases correspond to type E Dynkin diagrams, with 12, 18 and 30 being the Coxeter numbers of E_6 , E_7 and E_8).

The problem with these three exceptional types was that the classification of integral representations of the group algebra of the corresponding Weyl group predicted the existence of additional “type E ” simple transitive 2-representations. These additional simple transitive 2-representations were constructed later in [MT] (relying on ideas from [KS, AT]), using a presentation for Soergel bimodules given in [EI]. This method differs conceptually from the one used in [MM5, KMMZ] (and the other papers mentioned above) where 2-representations were constructed as “subquotients” of the so-called *principal 2-representations*. At the moment, there is no universal algorithm which would allow one to move between the 2-representations constructed using these two different methods.

The main motivation for the present paper is to develop some techniques to reinterpret the results of [MT] in the framework of the approach of [MM4, KMMZ]. For this we extend to our setup and further develop the ideas of [Os, EO, ENO, EGNO] which study 2-representations of certain tensor categories using algebra objects in these tensor categories and the module categories associated to these algebra objects.

An interesting example is given by the semisimplified quotient of $U_q(\mathfrak{sl}_2)\text{-mod}$, where q is a primitive complex even root of unity. In this case, as it is shown in [KO, Os], there are three families of algebra objects. These correspond to the simply laced Dynkin diagrams, just like the simple transitive 2-representations of Soergel bimodules in dihedral types.

In the present paper, we show that, for a given fiat 2-category \mathcal{C} (in the sense of [MM1]), there is a bijection between the equivalence classes of simple transitive 2-representations of \mathcal{C} and the Morita–Takeuchi equivalence classes of simple coalgebra 1-morphisms in $\underline{\mathcal{C}}$, the injective abelianization of \mathcal{C} , see Theorem 4.7. Dually, we also show that there is a bijection between the equivalence classes of simple transitive 2-representations of \mathcal{C} and the Morita equivalence classes of algebra 1-morphisms in $\overline{\mathcal{C}}$, the projective abelianization of \mathcal{C} , see Corollary 4.8. Here the Morita(–Takeuchi) theory for (co)algebra 1-morphisms in \mathcal{C} is a direct generalization of the classical Morita(–Takeuchi) theory for (co)algebras [Mo, Ta], as we explain in Section 5. (This is not to be confused with the results of [MM4].)

Our results extend and generalize some of the results in [Os, EO, ENO, EGNO]. However, there are some essential difficulties due to the fact that our setup differs from the one studied in [Os, EO, ENO, EGNO]: For example, the latter references work mostly with abelian monoidal categories (with some extra structure), while the categories we consider are additive, but almost never abelian. One of the manifestations of this difficulty is our definition of *internal homs* in Section 4.1 which is, in some sense, dual to the one used in [EGNO]. Indeed, it turns out that the internal hom defined in [EGNO, Definition 7.9.2] does not have the necessary properties which would allow one to develop a useful theory in our setup. See also Remark 4.1.

We also give several (classes of) examples. One of them explains the relation between the two aforementioned ADE classifications. For this we crucially rely on [EI]. That is, Elias’ results show that there is a 2-functor between the semisimplified quotient

of $U_q(\mathfrak{sl}_2)\text{-mod}$, with q as above, and the small quotient of the 2-category of singular Soergel bimodules of dihedral type. We show that this 2-functor gives the link between the two ADE classifications, using our relation between simple transitive 2-representations and algebra 1-morphisms, see Section 7.

Remark 1.1. It is worth emphasizing that the approach using (co)algebra 1-morphism does not seem to be very helpful for the classification of 2-representations, because the classification of (co)algebra 1-morphisms looks like a very hard problem in general. However, this method is quite helpful if one would like to check existence of some 2-representations, since this can be reformulated into the problem of checking that certain 1-morphisms have an additional structure of a (co)algebra 1-morphism. This is sometimes quite easy, e.g. the type A and D algebra 1-morphisms for the semisimplified quotient of $U_q(\mathfrak{sl}_2)\text{-mod}$ decategorify to idempotents in the Grothendieck group – and this basically fixes the algebra structure, cf. Remark 7.5.

The paper is organized as follows: Section 2 contains preliminaries on 2-representations. Section 3 introduces a new version of abelianization for finitary 2-categories and compares it to the previous versions defined in [MM1]. This new abelianization is essential in the rest of the paper as it significantly simplifies arguments related to abelianization of 2-representations. Section 4 contains our main results mentioned above. Section 5 establishes an analogue of Morita(–Takeuchi) theory in our setup. Finally, Sections 6 and 7 deal with some explicit examples and applications, which include 2-categories of projective functors for finite dimensional algebras and 2-categories of Soergel bimodules of dihedral type.

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2. SOME RECOLLECTIONS OF 2-REPRESENTATION THEORY

2.1. Basic notation and conventions. We fix an algebraically closed field k . All of our (2-)categories and (2-)functors etc. will be k -linear unless stated otherwise.

A *2-category* is a category enriched over the category of all (small) categories. That is, a 2-category \mathcal{C} consists of a collection of objects denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ etc.; for each pair (\mathbf{i}, \mathbf{j}) of objects, a small category $\mathcal{C}(\mathbf{i}, \mathbf{j})$ consisting of a set of 1-morphisms, whose elements will be denoted by F, G, H etc., and, for each pair (F, G) of 1-morphisms in a fixed $\mathcal{C}(\mathbf{i}, \mathbf{j})$, a set $\text{Hom}_{\mathcal{C}(\mathbf{i}, \mathbf{j})}(F, G)$ of 2-morphisms (we also write $\text{Hom}_{\mathcal{C}}(F, G)$ etc. for short), whose elements we call α, β, γ etc. For any $\mathbf{i} \in \mathcal{C}$, we denote by $\mathbb{1}_{\mathbf{i}}$ the identity 1-morphism in $\mathcal{C}(\mathbf{i}, \mathbf{i})$. For a 1-morphism F , we write id_F for the corresponding identity 2-morphism in $\text{Hom}_{\mathcal{C}(\mathbf{i}, \mathbf{j})}(F, F)$. Moreover, we use the symbol

\circ for composition of 1-morphisms (but often omit it), \circ_0 for horizontal composition and \circ_1 for vertical composition of 2-morphisms. (For more background on abstract 2-categories, see e.g. [ML, Le].)

The 2-category \mathcal{C} is called *\mathbb{k} -linear* if $\text{Hom}_{\mathcal{C}(\mathbf{i}, \mathbf{j})}(F, G)$ is a \mathbb{k} -vector space, for all (F, G) , and if horizontal and vertical compositions are \mathbb{k} -bilinear operations.

Moreover, we will also meet the notion of a *bicategory*. We do not need bicategories often in this paper and refer the reader to [Be, ML, Le] for details. The example to keep in mind are (non-strict) monoidal categories. The most important, for us, fact to recall about bicategories is that they can always be *strictified*: any bicategory is weakly equivalent to a 2-category, see e.g. [Be] or [Le, Theorem 2.3].

2.2. Finitary categories and 2-categories. An additive \mathbb{k} -linear category is called *finitary* if it has split idempotents, only finitely many isomorphism classes of indecomposable objects and the morphism sets are finite dimensional \mathbb{k} -vector spaces. We write $\mathfrak{A}_{\mathbb{k}}^f$ for the 2-category which has

- finitary additive \mathbb{k} -linear categories as objects;
- \mathbb{k} -linear (hence, additive) functors as 1-morphisms;
- natural transformations of functors as 2-morphisms.

Then we say a 2-category \mathcal{C} is *finitary* provided

- it only has finitely many objects;
- for each pair (\mathbf{i}, \mathbf{j}) of objects, the category $\mathcal{C}(\mathbf{i}, \mathbf{j})$ is in $\mathfrak{A}_{\mathbb{k}}^f$;
- horizontal composition is additive and \mathbb{k} -linear;
- the identity 1-morphism $\mathbb{1}_{\mathbf{i}}$ is indecomposable for every $\mathbf{i} \in \mathcal{C}$.

2.3. Fiat 2-categories. For any 2-category \mathcal{C} , we consider the 2-category $\mathcal{C}^{\text{co,op}}$, which is obtained from \mathcal{C} by reversing both 1- and 2-morphisms.

We say that a finitary 2-category \mathcal{C} is *weakly fiat* if \mathcal{C} is endowed with a weak equivalence $*$: $\mathcal{C} \rightarrow \mathcal{C}^{\text{co,op}}$ such that, for any pair (\mathbf{i}, \mathbf{j}) of objects and every 1-morphism $F \in \mathcal{C}(\mathbf{i}, \mathbf{j})$, there are 2-morphisms $\alpha: F \circ F^* \rightarrow \mathbb{1}_{\mathbf{j}}$ and $\beta: \mathbb{1}_{\mathbf{i}} \rightarrow F^* \circ F$ satisfying $\alpha_F \circ_1 F(\beta) = \text{id}_F$ and $F^*(\alpha) \circ_1 \beta_{F^*} = \text{id}_{F^*}$. Note that α and β define an adjunction between F and F^* . We therefore call them *adjunction 2-morphisms*.

We denote by $*F$ the image of a 1-morphism F under an inverse to $*$.

If $*$ is a weak involution, we say that \mathcal{C} is *fiat*. (We refer the reader to [MM1, MM2, MM6] for details about (weakly) fiat categories.)

2.4. 2-representations. Let \mathcal{C} be a finitary 2-category. By a *2-representation* of \mathcal{C} we mean a strict 2-functor from \mathcal{C} to the 2-category of (small) categories.

Such a 2-representation is called a *finitary 2-representation* if it is a strict 2-functor from \mathcal{C} to $\mathfrak{A}_{\mathbb{k}}^f$. We usually denote 2-representations by $\mathbf{M}, \mathbf{N}, \dots$. For any fixed $\mathbf{i} \in \mathcal{C}$, we use the symbol $\mathbf{P}_{\mathbf{i}}$ for the \mathbf{i} -th *principal* 2-representation $\mathcal{C}(\mathbf{i}, -)$.

All finitary 2-representations of \mathcal{C} form a 2-category whose 1-morphisms are 2-natural transformations and whose 2-morphisms are modifications (more details are given in [Le, MM3]). Moreover, we say that two 2-representations \mathbf{M} and \mathbf{N} of \mathcal{C} are *equivalent*, if there exists a 2-natural transformation $\Phi: \mathbf{M} \rightarrow \mathbf{N}$ which induces an equivalence of categories, for each object i .

2.5. 2-ideals. A semicategory is a collection of objects and morphisms satisfying the axioms of a category except for the existence of identity morphisms. Similarly, a 2-semicategory is a category enriched over semicategories.

Given any 2-category \mathcal{C} , a *left 2-ideal* \mathcal{I} of \mathcal{C} is a 2-semicategory, which has the same objects as \mathcal{C} and in which, for each pair (i, j) of objects, $\mathcal{I}(i, j)$ is an ideal in $\mathcal{C}(i, j)$, closed under left horizontal multiplication with both 1- and 2-morphisms in \mathcal{C} . Similarly, one defines *right 2-ideals* and *two-sided 2-ideals* (which we also just call *2-ideals*). An important class of left 2-ideals in \mathcal{C} is given by the i -th principal 2-representations \mathbf{P}_i .

If \mathbf{M} is a 2-representation \mathcal{C} , an *ideal* \mathbf{I} in \mathbf{M} is the data of an ideal $\mathbf{I}(i)$ in $\mathbf{M}(i)$, for each $i \in \mathcal{C}$, which is stable under the action of \mathcal{C} .

2.6. Simple transitive 2-representations. Let \mathcal{C} be a finitary 2-category. We call a finitary 2-representation \mathbf{M} of \mathcal{C} *transitive* if, for every object $i \in \mathcal{C}$ and every non-zero object $X \in \mathbf{M}(i)$, the 2-subrepresentation $\mathbf{G}_{\mathbf{M}}(X)$ of \mathbf{M} is equivalent to \mathbf{M} . Here $\mathbf{G}_{\mathbf{M}}(X)$ is the additive closure $\text{add}(\{\mathbf{M}(F)X\})$, where F runs over all 1-morphisms of \mathcal{C} . In what follows we will often use the module (action) notation $F X$ instead of the representation notation $\mathbf{M}(F)X$.

A transitive 2-representation \mathbf{M} has a unique maximal ideal \mathbf{I} not containing any identity morphisms other than that of the zero object (cf. [MM5, Lemma 4]). If $\mathbf{I} = 0$, then \mathbf{M} is said to be *simple transitive*. In general, the quotient $\widehat{\mathbf{M}}$ of \mathbf{M} by \mathbf{I} is simple transitive and is called the *simple transitive quotient* of \mathbf{M} .

2.7. Combinatorics of 1-morphisms. Recall that a multisemigroup is a pair consisting of a set \mathcal{S} and an associative multivalued operation from $\mathcal{S} \times \mathcal{S}$ to the set of subsets of \mathcal{S} .

Let \mathcal{C} be a finitary 2-category and denote by $\mathcal{S}(\mathcal{C})$ the set of isomorphism classes of indecomposable 1-morphisms in \mathcal{C} . This has the structure of a multisemigroup by [MM2, Section 3], which comes equipped with several preorders.

For two 1-morphisms F and G , we have $G \geq_L F$ in the *left preorder* if there exists a 1-morphism H such that G occurs, up to isomorphism, as a direct summand in $H \circ F$. An equivalence class for this preorder is called a *left cell*. Similarly, we define the *right* and *two-sided* preorders \geq_R and \geq_J , and the corresponding *right* and *two-sided* cells.

Observe that \geq_L defines a partial order on the set of left cells, and, similarly, \geq_R and \geq_J define partial orders on the sets of right cells and two-sided cells.

Note that, if \mathcal{C} is weakly fiat, then both $F \mapsto F^*$ and $F \mapsto *F$ induce isomorphisms between the partially ordered sets $(\mathcal{S}(\mathcal{C}), \leq_L)$ and $(\mathcal{S}(\mathcal{C}), \leq_R)$.

2.8. Cell 2-representations. Let \mathcal{C} be a finitary 2-category. Consider a 2-representation \mathbf{M} of \mathcal{C} such that, for each object $\mathbf{i} \in \mathcal{C}$, the category $\mathbf{M}(\mathbf{i})$ is additive and idempotent complete. Let I be a subset of objects in \mathcal{C} . Given any collection $X_i \in \mathbf{M}(\mathbf{i}_i)$ of objects, where $i \in I$, define $\mathbf{G}_{\mathbf{M}}(\{X_i \mid i \in I\})$ as in Section 2.6. This becomes a 2-subrepresentation of \mathbf{M} by restriction.

For any left cell \mathcal{L} in \mathcal{C} , there exists an object $\mathbf{i} = \mathbf{i}_{\mathcal{L}} \in \mathcal{C}$ such that the domain of every 1-morphism in \mathcal{L} is \mathbf{i} . Therefore it makes sense to define the 2-representation $\mathbf{N} = \mathbf{G}_{\mathbf{P}_{\mathbf{i}}}(\mathcal{L})$, which, by [MM5, Lemma 3], has a unique maximal ideal \mathbf{I} not containing $\text{id}_{\mathbf{F}}$, for any $\mathbf{F} \in \mathcal{L}$. We call the quotient $\mathbf{C}_{\mathcal{L}} = \mathbf{N}/\mathbf{I}$ the *(additive) cell 2-representation* of \mathcal{C} associated to \mathcal{L} .

3. SEVERAL VERSIONS OF ABELIANIZATION

3.1. Classical abelianization. Let \mathcal{A} be a finitary category. Recall, see e.g. [Fr], that the (diagrammatic) *injective abelianization* $\underline{\underline{\mathcal{A}}}$ of \mathcal{A} is defined as follows:

- objects of $\underline{\underline{\mathcal{A}}}$ are diagrams $X \xrightarrow{f} Y$ over \mathcal{A} ;
- morphisms in $\underline{\underline{\mathcal{A}}}$ are equivalence classes of solid commutative diagrams of the form (i.e. without the diagonal dashed arrow)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & \swarrow q & \downarrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

modulo the ideal generated by those diagrams for which there is a “homotopy” q as shown by the dashed arrow such that $g = qf$;

- identity morphisms are given by diagrams in which both g and h are the identities;
- composition is given by the vertical composition of diagrams.

The category $\underline{\underline{\mathcal{A}}}$ is abelian. In particular, the object $X \xrightarrow{f} Y$ has an embedding into $X \xrightarrow{0} 0$, its injective hull, and this embedding is a kernel of f . Moreover, the category $\underline{\underline{\mathcal{A}}}$ is equivalent (here and further in similar situations: as a \mathbf{k} -linear category) to the category of *right* finite dimensional \mathcal{A} -modules. The original category \mathcal{A} embeds into $\underline{\underline{\mathcal{A}}}$ via

$$X \mapsto X \longrightarrow 0 \quad \text{and} \quad f: X \rightarrow Y \mapsto \begin{array}{ccc} X & \longrightarrow & 0 \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & 0 \end{array}$$

and this embedding induces an equivalence between \mathcal{A} and the full subcategory of injective objects in $\underline{\underline{\mathcal{A}}}$.

The (diagrammatic) *projective abelianization* $\overline{\overline{\mathcal{A}}}$ is defined in the dual way, see e.g. [Fr] or [MM1, Section 3.1].

3.2. A different version of injective abelianization. Let \mathcal{A} be a finitary category. Now we define a slightly different version $\underline{\mathcal{A}}$ of the (diagrammatic) *injective abelianization* of \mathcal{A} in the following way:

- objects in $\underline{\mathcal{A}}$ are tuples of the form $(X, k, Y_i, f_i)_{i=1}^\infty$, where $k \in \mathbb{Z}_{\geq 0}$, X and Y_i are objects in \mathcal{A} , and $f_i: X \rightarrow Y_i$ are morphisms in \mathcal{A} , with the additional requirement that $Y_i = 0$ for all $i > k$;
- morphisms in $\underline{\mathcal{A}}$ from $(X, k, Y_i, f_i)_{i=1}^\infty$ to $(X', k', Y'_i, f'_i)_{i=1}^\infty$ are equivalence classes of tuples $(g, h_{i,j})_{i,j=1}^\infty$, where $g: X \rightarrow X'$ and $h_{i,j}: Y_i \rightarrow Y'_j$ are morphisms in \mathcal{A} such that $f'_i g = \sum_j h_{j,i} f_j$, for each i , modulo the equivalence relation given by the homotopy relation spanned by those tuples $(g, h_{i,j})$ for which there exist $q_i: Y_i \rightarrow X'$ such that $\sum_i q_i f_i = g$;
- identity morphisms are given by tuples $(g, h_{i,j})_{i,j=1}^\infty$ in which $h_{i,j} = 0$, if $i \neq j$, and the remaining morphism are the identities;
- composition of the tuple $(g, h_{i,j})_{i,j=1}^\infty$ followed by the tuple $(g', h'_{i,j})_{i,j=1}^\infty$ is defined as the tuple $(g'g, \sum_k h'_{k,j} h_{i,k})_{i,j=1}^\infty$.

One should think about $\underline{\mathcal{A}}$ as a version of $\underline{\mathcal{A}}$ with “multiple arrows” from the left object to the “multiple objects” on the right:

$$\begin{array}{ccc} & & Y_1 \\ & \nearrow^{f_1} & \\ X & \xrightarrow{f_2} & Y_2 \\ & \searrow_{f_i} & \vdots \end{array}$$

These multiple arrows are indexed by non-negative integers, only finitely many of these arrows go to non-zero objects and k is a fixed explicit bound saying that after it arrows must terminate at the zero objects.

The category $\underline{\mathcal{A}}$ is additive, with \oplus given by

$$(X, k, Y_i, f_i)_{i=1}^\infty \oplus (X', k', Y'_i, f'_i)_{i=1}^\infty = (X \oplus X', \max(k, k'), Y_i \oplus Y'_i, f_i \oplus f'_i)_{i=1}^\infty$$

with the evident definition on morphisms. Further, the assignment

$$(X, k, Y_i, f_i)_{i=1}^\infty \mapsto X \xrightarrow{\oplus_i f_i} \oplus_i Y_i$$

provides an equivalence between $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}}$ which, in particular, implies that $\underline{\mathcal{A}}$ is abelian. Indeed, an inverse equivalence is given, for example, by sending $X \xrightarrow{f} Y$ to $(X, 1, Y_i, f_i)_{i=1}^\infty$, where $Y_1 = Y$, $f_1 = f$, $Y_i = 0$ and $f_i = 0$, for $i > 1$.

The original category \mathcal{A} embeds into $\underline{\mathcal{A}}$ via

$$X \mapsto (X, 0, 0, 0) \quad \text{and} \quad f: X \rightarrow Y \mapsto (f, 0).$$

The main point of the definition of $\underline{\mathcal{A}}$ is that the “multiple objects” will give us a chance to bookkeep some explicit direct sum constructions in the next sections.

The (diagrammatic) *projective abelianization* $\overline{\mathcal{A}}$ is defined dually.

3.3. Abelianization of finitary 2-categories. The construction presented here rectifies some problems with strictness pointed out in [MM1, Section 3.5].

Let \mathcal{C} be a finitary 2-category. Consider the (diagrammatic) *injective abelianization* $\underline{\mathcal{C}}$ of \mathcal{C} defined as follows:

- $\underline{\mathcal{C}}$ has the same objects as \mathcal{C} ;
- $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j}) = \underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$ (using the above notation for injective abelianization);
- composition of 1-morphisms is defined as follows:

$$(F, k, G_i, \alpha_i)_{i=1}^\infty \circ (F', k', G'_i, \alpha'_i)_{i=1}^\infty = (FF', k + k', H_i, \beta_i)_{i=1}^\infty,$$

where

$$H_i = \begin{cases} F \circ G'_i, & i = 1, 2, \dots, k'; \\ G_{i-k'} \circ F', & i = k' + 1, k' + 2, \dots, k' + k; \\ 0, & \text{else}; \end{cases}$$

$$\beta_i = \begin{cases} \text{id}_F \circ_0 \alpha'_i, & i = 1, 2, \dots, k'; \\ \alpha_{i-k'} \circ_0 \text{id}_{F'}, & i = k' + 1, k' + 2, \dots, k' + k; \\ 0, & \text{else}. \end{cases}$$

- identity 1-morphisms are tuples $(\mathbb{1}_{\mathbf{i}}, 0, 0, 0)$, for $\mathbf{i} \in \mathcal{C}$;
- horizontal composition of 2-morphisms is defined component-wise.

For $\mathbf{i}, \mathbf{j} \in \mathcal{C}$, mapping F to $(F, 0, 0, 0)$ gives rise to an equivalence between $\mathcal{C}(\mathbf{i}, \mathbf{j})$ and the full subcategory of injective objects in $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$ which we will use to identify these two 2-categories. Note that this also realizes \mathcal{C} as a 2-subcategory of $\underline{\mathcal{C}}$.

The (diagrammatic) *projective abelianization* $\overline{\mathcal{C}}$ is defined dually.

Remark 3.1. (a) We note that, generally, even in the case when \mathcal{C} is a fiat 2-category, neither $\underline{\mathcal{C}}$ nor $\overline{\mathcal{C}}$ have adjunction 2-morphisms or even a weak involution (see also Remark 4.1). In fact, if \mathcal{C} is fiat, the weak involution on \mathcal{C} extends to a contravariant biequivalence between $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$, which changes the direction of both 1- and 2-morphisms.

(b) In general, we do not know what kind of exactness properties the horizontal composition bifunctor \circ on $\underline{\mathcal{C}}$ might have. At the same time, in the case when \mathcal{C} is a fiat 2-category, both the left and the right regular actions of \mathcal{C} on $\underline{\mathcal{C}}$ are, automatically, given by exact functors.

3.4. Abelianization of finitary 2-representations. Let \mathcal{C} be a finitary 2-category and \mathbf{M} a finitary 2-representation of \mathcal{C} . Then the (diagrammatic) *injective abelianization* $\underline{\mathbf{M}}$ of \mathbf{M} is defined by $\underline{\mathbf{M}}(\mathbf{i}) = \underline{\mathbf{M}}(\mathbf{i})$, for $\mathbf{i} \in \mathcal{C}$ (here again we use the above notation for injective abelianization). Note that $\underline{\mathbf{M}}$ has the structure of a 2-representation of \mathcal{C} given by the component-wise action.

Moreover, $\underline{\mathbf{M}}$ has the natural structure of a 2-representation of $\underline{\mathcal{C}}$ with the action defined on objects as follows:

$$(F, k, G_i, \alpha_i)_{i=1}^\infty \circ (M, k', N_i, f_i)_{i=1}^\infty = (FM, k + k', H_i, g_i)_{i=1}^\infty,$$

where

$$H_i = \begin{cases} F N_i, & i = 1, 2, \dots, k'; \\ G_i M, & i = k' + 1, k' + 2, \dots, k' + k; \\ 0, & \text{else}; \end{cases}$$

$$g_i = \begin{cases} F f_i, & i = 1, 2, \dots, k'; \\ (\alpha_{i-k'})_M, & i = k' + 1, k' + 2, \dots, k' + k; \\ 0, & \text{else}; \end{cases}$$

and with the component-wise action on morphisms.

Similarly to the above, the canonical embedding of \mathbf{M} into $\underline{\mathbf{M}}$ is a morphism of 2-representations of \mathcal{C} and provides an equivalence between \mathbf{M} and the 2-representation of \mathcal{C} given by the action of \mathcal{C} on injective objects of $\coprod_{i \in \mathcal{C}} \underline{\mathbf{M}}(i)$.

As usual, the *projective abelianization* $\overline{\mathbf{M}}$ is defined dually.

4. FINITARY 2-REPRESENTATIONS VIA (CO)ALGEBRA 1-MORPHISMS

This section is inspired by [EGNO, Chapter 7]. The main goal is to provide a generalization of [EGNO, Theorem 7.10.1, Corollary 7.10.5] (see also [Os, EO, ENO]).

Remark 4.1. The framework of [EGNO] is that of a tensor category, where all objects have duals and which is assumed to be abelian. In our situation, we have a 2-category \mathcal{C} instead of a tensor category, moreover, we assume that \mathcal{C} is enriched over additive and not necessarily abelian categories. The existence of dual objects in the language of [EGNO] translates into the existence of adjoint 1-morphisms (and, in particular, adjunction 2-morphisms) in our setup, as described in Section 2.3. If our 2-category \mathcal{C} is fiat, these adjunctions exist. However, in passing to the abelianization $\underline{\mathcal{C}}$, we lose the existence of adjunction 2-morphisms. Consequently, we are forced to use a construction which is dual to the one in [EGNO, Section 7.9] (as some important properties fail to hold for the direct generalization of the construction therein).

Before starting, we point out that the abstract notions of (co)algebra objects and their (co)module categories from [EGNO, Section 7.8] generalize immediately to our setting. The only difference is that instead of (co)algebra objects in monoidal categories, we consider *(co)algebra 1-morphisms* in 2-categories.

4.1. Internal homs. Recall that left exact functors between module categories are determined uniquely, up to isomorphism, by their action on the category of injective modules, see for example [Ba, Chapter II, §2] and the dual version of it. In particular, given an algebra A , any additive functor from its category of injective modules $A\text{-inj}$ to an abelian category \mathcal{B} extends uniquely, up to isomorphism, to a left exact functor from its category of modules $A\text{-mod}$ to \mathcal{B} , and this extension is natural with respect to natural transformations of functors. This observation motivates the following construction.

Let \mathcal{C} be a fiat 2-category and \mathbf{M} a transitive 2-representation of \mathcal{C} . For $i, j \in \mathcal{C}$, $M \in \mathbf{M}(i)$ and $N \in \mathbf{M}(j)$, consider the unique, up to isomorphism, left exact functor

from $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$ to the category of finite dimensional k -vector spaces given by

$$(1) \quad F \mapsto \text{Hom}_{\mathbf{M}(\mathbf{j})}(M, F N), \quad \text{for } F \in \mathcal{C}(\mathbf{i}, \mathbf{j}).$$

Uniqueness is due to the equivalence between $\mathcal{C}(\mathbf{i}, \mathbf{j})$ and the full subcategory of injective objects in $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$ as in Section 3.3.

Being left exact, the functor in (1) is representable, that is, there exists a 1-morphism $\underline{\text{Hom}}(N, M) \in \underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$, unique up to isomorphism, with an isomorphism

$$(2) \quad \text{Hom}_{\mathbf{M}(\mathbf{j})}(M, F N) \cong \text{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})}(\underline{\text{Hom}}(N, M), F), \quad \text{for any } F \in \mathcal{C}(\mathbf{i}, \mathbf{j}).$$

The 1-morphism $\underline{\text{Hom}}(N, M)$ is called the *internal hom from N to M* .

In fact, the isomorphism from (2) also exists for all $F \in \underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$:

Lemma 4.2. *There is an isomorphism*

$$(3) \quad \text{Hom}_{\mathbf{M}(\mathbf{j})}(M, F N) \cong \text{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})}(\underline{\text{Hom}}(N, M), F), \quad \text{for any } F \in \underline{\mathcal{C}}(\mathbf{i}, \mathbf{j}).$$

Proof. As the canonical embedding of $\mathcal{C}(\mathbf{i}, \mathbf{j})$ into the subcategory of injective objects of $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})$ is an equivalence, see Section 3.3, the claim follows from (2) by standard arguments using left exactness and the Five Lemma. \square

Lemma 4.3. *If $M = N$, then $\mathbf{i} = \mathbf{j}$ and $A^N = \underline{\text{Hom}}(N, N)$ has the structure of a coalgebra 1-morphism in $\underline{\mathcal{C}}(\mathbf{i}, \mathbf{i})$.*

Proof. We have the coevaluation map

$$\text{coev}_{N, N}: N \rightarrow \underline{\text{Hom}}(N, N) N$$

given by the image of $\text{id}_{\underline{\text{Hom}}(N, N)}$ under (the special case of) the isomorphism

$$\text{Hom}_{\mathbf{M}(\mathbf{i})}(N, \underline{\text{Hom}}(N, N) N) \cong \text{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{i})}(\underline{\text{Hom}}(N, N), \underline{\text{Hom}}(N, N))$$

in (3). Following the arguments in [EGNO, Equation (7.29)], this gives rise to the comultiplication morphism

$$\underline{\text{Hom}}(N, N) \rightarrow \underline{\text{Hom}}(N, N) \circ \underline{\text{Hom}}(N, N).$$

Similarly to [EGNO, Equation (7.30)], the counit map is defined using the image of id_N under (the special case of) the isomorphism

$$\text{Hom}_{\mathbf{M}(\mathbf{i})}(N, \mathbb{1}_{\mathbf{i}} N) \cong \text{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{i})}(\underline{\text{Hom}}(N, N), \mathbb{1}_{\mathbf{i}})$$

in (2). Finally, similarly to [EGNO, Section 7.9], coassociativity and the other axioms are checked by direct computation. \square

We note that existence of adjoint 1-morphisms (and adjunction 2-morphisms) for the (analog of the) whole of $\underline{\mathcal{C}}$ is not used in [EGNO, Equations (7.29) and (7.30)].

4.2. The categories of comodules. Let $\text{comod}_{\mathcal{C}}(A^N)$ denote the category of right A^N -comodule 1-morphisms in $\coprod_{j \in \mathcal{C}} \mathcal{C}(\mathbf{i}, j)$ and $\text{inj}_{\mathcal{C}}(A^N)$ denote the subcategory of injective objects in $\text{comod}_{\mathcal{C}}(A^N)$. Then we have the functor

$$(4) \quad \begin{aligned} \Theta: \coprod_{j \in \mathcal{C}} \mathbf{M}(j) &\rightarrow \text{comod}_{\mathcal{C}}(A^N), \\ M &\mapsto \underline{\text{Hom}}(N, M), \end{aligned}$$

with the evident assignment for morphisms.

Lemma 4.4. *The functor Θ (weakly) commutes with the action of \mathcal{C} and defines a morphism of 2-representations.*

Proof. We need to show that

$$(5) \quad \underline{\text{Hom}}(N, F M) \cong F \underline{\text{Hom}}(N, M),$$

for any 1-morphism F in $\mathcal{C}(\mathbf{i}, j)$. By the property of \mathcal{C} being fiat, (2) and uniqueness of the representing object up to isomorphism, we have

$$\begin{aligned} \text{Hom}_{\mathcal{C}(\mathbf{i}, j)}(\underline{\text{Hom}}(N, F M), G) &\cong \text{Hom}_{\mathbf{M}(j)}(F M, G N) \\ &\cong \text{Hom}_{\mathbf{M}(i)}(M, F^* \circ G N) \\ &\cong \text{Hom}_{\mathcal{C}(\mathbf{i}, i)}(\underline{\text{Hom}}(N, M), F^* \circ G) \\ &\cong \text{Hom}_{\mathcal{C}(\mathbf{i}, j)}(F \circ \underline{\text{Hom}}(N, M), G), \end{aligned}$$

for any $G \in \mathcal{C}(\mathbf{i}, j)$, and the first statement follows.

For the second, we need to check coherence, in other words, we need to check that for any $G \in \mathcal{C}(\mathbf{i}, j)$, $F \in \mathcal{C}(j, k)$, $H \in \mathcal{C}(\mathbf{i}, k)$, the composite of isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}(\mathbf{i}, k)}(\underline{\text{Hom}}(N, FGM), H) &\xrightarrow{\phi_1} \text{Hom}_{\mathbf{M}(k)}(FGM, HN) \xrightarrow{\phi_2} \text{Hom}_{\mathbf{M}(i)}(M, (FG)^* HN) \\ &\xrightarrow{\phi_3} \text{Hom}_{\mathcal{C}(\mathbf{i}, i)}(\underline{\text{Hom}}(N, M), (FG)^* H) \xrightarrow{\phi_4} \text{Hom}_{\mathcal{C}(\mathbf{i}, k)}(FG \underline{\text{Hom}}(N, M), H) \end{aligned}$$

is equal to the composite of isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}(\mathbf{i}, k)}(\underline{\text{Hom}}(N, FGM), H) &\xrightarrow{\psi_1} \text{Hom}_{\mathbf{M}(k)}(FGM, HN) \xrightarrow{\psi_2} \text{Hom}_{\mathbf{M}(j)}(GM, F^* HN) \\ &\xrightarrow{\psi_3} \text{Hom}_{\mathcal{C}(\mathbf{i}, j)}(\underline{\text{Hom}}(N, GM), F^* H) \xrightarrow{\psi_4} \text{Hom}_{\mathbf{M}(j)}(GM, F^* HN) \\ &\xrightarrow{\psi_5} \text{Hom}_{\mathbf{M}(i)}(M, G^* F^* HN) \xrightarrow{\psi_6} \text{Hom}_{\mathcal{C}(\mathbf{i}, i)}(\underline{\text{Hom}}(N, M), G^* F^* H) \\ &\xrightarrow{\psi_7} \text{Hom}_{\mathcal{C}(\mathbf{i}, j)}(G \underline{\text{Hom}}(N, M), F^* H) \xrightarrow{\psi_8} \text{Hom}_{\mathcal{C}(\mathbf{i}, k)}(FG \underline{\text{Hom}}(N, M), H). \end{aligned}$$

Since $\phi_1 = \psi_1$ and ψ_3 and ψ_4 are mutual inverses, it suffices to check that $\phi_4\phi_3\phi_2 = \psi_8\psi_7\psi_6\psi_5\psi_2$, that is, that the solid part of the following diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{M}(\mathbf{k})}(\mathrm{FG}M, \mathrm{HN}) & \xrightarrow{\psi_2} & \mathrm{Hom}_{\mathbf{M}(\mathbf{j})}(\mathrm{GM}, \mathrm{F}^*\mathrm{HN}) \\
\downarrow \phi_2 & & \downarrow \psi_5 \\
\mathrm{Hom}_{\mathbf{M}(\mathbf{i})}(M, (\mathrm{FG})^*\mathrm{HN}) & \xleftarrow{\quad \text{dashed} \quad} & \mathrm{Hom}_{\mathbf{M}(\mathbf{i})}(M, \mathrm{G}^*\mathrm{F}^*\mathrm{HN}) \\
\downarrow \phi_3 & & \downarrow \psi_6 \\
\mathrm{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{i})}(\underline{\mathrm{Hom}}(N, M), (\mathrm{FG})^*\mathrm{H}) & \xleftarrow{\quad \text{dashed} \quad} & \mathrm{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{i})}(\underline{\mathrm{Hom}}(N, M), \mathrm{G}^*\mathrm{F}^*\mathrm{H}) \\
\downarrow \phi_4 & & \downarrow \psi_7 \\
\mathrm{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{k})}(\mathrm{FG}\underline{\mathrm{Hom}}(N, M), \mathrm{H}) & \xleftarrow{\psi_8} & \mathrm{Hom}_{\underline{\mathcal{C}}(\mathbf{i}, \mathbf{j})}(\mathrm{G}\underline{\mathrm{Hom}}(N, M), \mathrm{F}^*\mathrm{H})
\end{array}$$

commutes. Uniqueness of adjoints up to unique isomorphism (cf. [EGNO, Proposition 2.10.5]) gives us a unique isomorphism $(\mathrm{FG})^* \cong \mathrm{G}^*\mathrm{F}^*$, which fills in the dashed arrows in the diagram, making the top and bottom squares commutative. Commutativity of the middle square is due to naturality of the isomorphism (3) from Lemma 4.2. Coherence of Θ and hence the claim that it is a morphism of 2-representations follows. \square

Lemma 4.5. *For any 1-morphism F in \mathcal{C} and any $X \in \mathrm{comod}_{\underline{\mathcal{C}}}(A^N)$, we have an isomorphism*

$$(6) \quad \mathrm{Hom}_{\mathrm{comod}_{\underline{\mathcal{C}}}(A^N)}(X, F A^N) \cong \mathrm{Hom}_{\underline{\mathcal{C}}}(X, F).$$

Proof. Postcomposing $\alpha: X \rightarrow F A^N$ with the map $F A^N \rightarrow F$ obtained by applying F to the counit morphism of A^N , we get a \mathbf{k} -linear map from the left-hand side to the right-hand side of (6).

Precomposing the evaluation of $\beta: X \rightarrow F$ at A^N with the map $X \rightarrow X A^N$ coming from the comodule structure on A^N , we get a \mathbf{k} -linear map from the right-hand side to the left-hand side of (6).

It is straightforward to check that these two maps are inverse to each other. \square

Lemma 4.6. *The functor Θ factors over the inclusion $\mathrm{inj}_{\underline{\mathcal{C}}}(A^N) \hookrightarrow \mathrm{comod}_{\underline{\mathcal{C}}}(A^N)$.*

Proof. If $M = F N$, for some 1-morphism F in \mathcal{C} , then, from (5), we have

$$\underline{\mathrm{Hom}}(N, F N) \cong F A^N.$$

Next we claim that A^N is injective in $\mathrm{comod}_{\underline{\mathcal{C}}}(A^N)$. From Lemma 4.5, it follows that

$$\mathrm{Hom}_{\mathrm{comod}_{\underline{\mathcal{C}}}(A^N)}(-, A^N) \cong \mathrm{Hom}_{\underline{\mathcal{C}}}(-, \mathbb{1}_i).$$

Hence, injectivity of A^N as a comodule reduces to injectivity of $\mathbb{1}_i$ as an object of $\underline{\mathcal{C}}$, which holds by construction of the injective abelianization, cf. Section 3.3.

Due to flatness of \mathcal{C} , it follows that $F A^N$ is injective in $\mathrm{comod}_{\underline{\mathcal{C}}}(A^N)$. By transitivity of \mathbf{M} , any M is isomorphic to a direct summand of some $F N$. The claim follows. \square

4.3. Transitive 2-representations and comodule categories. Recall that \mathcal{C} denotes a fiat 2-category and \mathbf{M} a transitive 2-representation of \mathcal{C} .

Theorem 4.7. *Let $N \in \mathbf{M}(\mathbf{i})$ be non-zero. Then the functor Θ gives rise to an equivalence of 2-representations of \mathcal{C} between $\underline{\mathbf{M}}$ and $\text{comod}_{\underline{\mathcal{C}}}(A^N)$. This equivalence restricts to an equivalence of 2-representations of \mathcal{C} between \mathbf{M} and $\text{inj}_{\underline{\mathcal{C}}}(A^N)$.*

Proof. For any objects of the form FN, GN in $\mathcal{M} = \coprod_{j \in \mathcal{C}} \mathbf{M}(j)$, we have

$$\begin{aligned} \text{Hom}_{\text{comod}_{\underline{\mathcal{C}}}(A^N)}(\Theta(FN), \Theta(GN)) &\cong \text{Hom}_{\text{comod}_{\underline{\mathcal{C}}}(A^N)}(F A^N, G A^N) \\ &\cong \text{Hom}_{\underline{\mathcal{C}}}(\underline{F} A^N, G) \\ &\cong \text{Hom}_{\underline{\mathcal{C}}}(A^N, F^* G) \\ &\cong \text{Hom}_{\mathcal{M}}(N, F^* G N) \\ &\cong \text{Hom}_{\mathcal{M}}(F N, G N), \end{aligned}$$

where the first isomorphism uses Lemma 4.4 and the second isomorphism uses Lemma 4.5. From transitivity of \mathbf{M} , we deduce that, for any $N_1, N_2 \in \mathcal{M}$, we have

$$\text{Hom}_{\text{comod}_{\underline{\mathcal{C}}}(A^N)}(\Theta(N_1), \Theta(N_2)) \cong \text{Hom}_{\mathcal{M}}(N_1, N_2).$$

The functor $\underline{\text{Hom}}(N, -)$ extends uniquely (up to isomorphism) to a left exact functor from $\underline{\mathcal{M}} = \coprod_{j \in \mathcal{C}} \mathbf{M}(j)$ to $\text{inj}_{\underline{\mathcal{C}}}(A^N)$, where the latter category is equivalent to $\text{comod}_{\underline{\mathcal{C}}}(A^N)$, as usual (cf. Section 3).

Considering injective resolutions of $N_1, N_2 \in \underline{\mathcal{M}}$ by objects in \mathcal{M} , using left exactness and arguments similar to the one in the proof of [EGNO, Theorem 7.10.1(2)], one checks that, indeed,

$$\text{Hom}_{\text{comod}_{\underline{\mathcal{C}}}(A^N)}(\Theta(N_2), \Theta(N_1)) \cong \text{Hom}_{\underline{\mathcal{M}}}(N_1, N_2),$$

for any $N_1, N_2 \in \underline{\mathcal{M}}$. Consequently, $\Theta: \underline{\mathcal{M}} \rightarrow \text{inj}_{\underline{\mathcal{C}}}(A^N)$ is full and faithful and it remains to show that it is essentially surjective.

To prove that Θ is essentially surjective, it suffices to show that Θ maps an injective cogenerator of $\underline{\mathbf{M}}$ to an injective cogenerator of $\text{comod}_{\underline{\mathcal{C}}}(A^N)$. In other words, it suffices to show that any injective A^N -comodule is isomorphic to a direct summand of a comodule of the form $F A^N$, for some 1-morphism F in \mathcal{C} . To prove the latter, we just need to show that any A^N -comodule injects into a comodule of the form $F A^N$, for some 1-morphism F . These claims are, basically, the injective versions of [EGNO, Exercises 7.8.14 and 7.8.15].

Let $X \in \text{comod}_{\underline{\mathcal{C}}}(A^N)$. Then the coaction map $X \mapsto X A^N$ is a homomorphism of A^N -comodules. This homomorphism is injective as its postcomposition with the evaluation $X A^N \rightarrow X$ at X of the counit morphism for A^N is the identity. Further, if X is of the form (Y, k, Z_i, α_i) , then we have a natural injection of $X A^N$ into $Y A^N$ given by the tuple $(\text{id}_Y \circ_0 \text{id}_{A^N}, 0)$. Now we note that the A^N -comodule $Y A^N$ has the necessary form. This shows that Θ is essentially surjective. As we have already established that it is full and faithful, the first claim of the theorem follows.

The second claim of the theorem follows from the first one and Lemma 4.6. \square

Different choices of N lead to Morita–Takeuchi equivalent coalgebra 1-morphisms, not to isomorphic ones, as we will explain in Corollary 5.2. For the remainder of this section, we fix some non-zero object N in $\underline{\mathcal{M}}$.

4.4. Transitive 2-representations and module categories. We get a “dual version” of Theorem 4.7 as well. For an algebra 1-morphism A in $\overline{\mathcal{C}}$, denote by $\text{mod}_{\overline{\mathcal{C}}}(A)$ the category of right A -module 1-morphisms in $\overline{\mathcal{C}}$ and by $\text{proj}_{\overline{\mathcal{C}}}(A)$ the subcategory of projective A -module 1-morphisms in $\overline{\mathcal{C}}$.

Corollary 4.8. *There exists an algebra 1-morphism A_N in $\overline{\mathcal{C}}$ and an equivalence of 2-representations of \mathcal{C} between $\overline{\mathbf{M}}$ and $\text{mod}_{\overline{\mathcal{C}}}(A_N)$, which restricts to an equivalence of 2-representations of \mathcal{C} between \mathbf{M} and $\text{proj}_{\overline{\mathcal{C}}}(A_N)$.*

Proof. The weak involution $*$ of the fiat category \mathcal{C} gives rise to a duality (on the level of both 1- and 2-morphisms) between $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$. This swaps coalgebra objects with algebra objects and comodules with modules. Therefore all claims follow from Theorem 4.7 using this duality. \square

Unfortunately, we do not see how to prove Corollary 4.8 directly, as in Theorem 4.7 we heavily rely on *left* exactness of all constructions, in particular, of the internal hom bifunctor.

Corollary 4.9. *If \mathbf{M} is simple transitive, then A_N is simple.*

Proof. If J is a non-trivial two-sided ideal of A_N , then all morphisms of the form $X\alpha$, where $\alpha \in J$ and $X \in \text{proj}_{\overline{\mathcal{C}}}(A_N)$, generate a non-trivial \mathcal{C} -stable ideal in $\text{proj}_{\overline{\mathcal{C}}}(A_N)$, contradicting simple transitivity of the action of \mathcal{C} on $\text{proj}_{\overline{\mathcal{C}}}(A_N)$. The fact that the ideal in question is non-trivial follows directly from nilpotency of the radical of any finite dimensional algebra. \square

Corollary 4.10. *For $\mathbf{i} \in \mathcal{C}$, consider the endomorphism 2-category \mathcal{A} of \mathbf{i} in \mathcal{C} (in particular, $\mathcal{A}(\mathbf{i}, \mathbf{i}) = \mathcal{C}(\mathbf{i}, \mathbf{i})$). Then there is a natural bijection between the equivalence classes of simple transitive 2-representations of \mathcal{A} and the equivalence classes of simple transitive 2-representations of \mathcal{C} having a non-trivial value at \mathbf{i} .*

Proof. We start by noting that \mathcal{A} inherits the structure of a fiat 2-category from \mathcal{C} . Now, consider a simple transitive 2-representation \mathbf{M} of \mathcal{C} such that $\mathbf{M}(\mathbf{i}) \neq 0$. Then we have the corresponding algebra 1-morphism A_N in $\overline{\mathcal{C}}$ given by Corollary 4.8. By construction, A_N is also in $\overline{\mathcal{A}}$ and, by Corollary 4.8, thus gives rise to simple transitive 2-representation of \mathcal{A} . (This 2-representation is just the restriction of \mathbf{M} to \mathcal{A} .) As any algebra 1-morphism in $\overline{\mathcal{A}}$ is, at the same time, an algebra 1-morphism in $\overline{\mathcal{C}}$, we obtain that the above correspondence is bijective. \square

4.5. Some remarks and a bonus observation.

Remark 4.11. Let \mathcal{C} be a fiat 2-category, \mathbf{M} a transitive 2-representation of \mathcal{C} and N an object in $\mathbf{M}(\mathbf{i})$. In all examples we know, there exists an algebra 1-morphism A in \mathcal{C} such that the algebra 1-morphism A_N is a quotient 1-morphism of A . However, we do not know whether this property holds in full generality.

The following bonus observation is inspired by [KMMZ, Theorem 2].

Proposition 4.12. *Assume that \mathcal{C} is a fiat 2-category, \mathbf{M} a simple transitive 2-representation of \mathcal{C} and N an object in $\mathbf{M}(\mathbf{i})$. Then A_N is self-injective.*

Before we give the proof of Proposition 4.12, recall the notion of an *apex* of a transitive 2-representation, given in [CM, Section 3], which is the maximal, with respect to the two-sided order, \mathcal{J} -cell which does not annihilate the 2-representation.

Proof. Let \mathcal{J} be the apex of \mathbf{M} . Analogously as in [KMMZ, Theorem 2], one can prove that 1-morphisms in \mathcal{J} send simple objects in $\overline{\mathbf{M}}$ to projective objects. A dual argument gives that 1-morphisms in \mathcal{J} send simple objects in $\overline{\mathbf{M}}$ to injective objects. The claim follows. \square

Remark 4.13. We could alternatively have used $\underline{\mathbf{M}}$ instead of $\overline{\mathbf{M}}$ in the proof of Proposition 4.12. As the referee pointed out to us, it is an interesting question whether $\underline{\mathbf{M}}$ and $\overline{\mathbf{M}}$ are equivalent (as abelian 2-representations). However, in full generality the answer is negative. For example, if \mathcal{C} is not a fiat 2-category, then $\underline{\mathbf{M}}$ and $\overline{\mathbf{M}}$ are not equivalent due to different sides of half-exactness for the 2-action of \mathcal{C} on them. On the other hand, thanks to the assertion of Proposition 4.12, such an equivalence seems to have a chance to be true in case \mathcal{C} is fiat and \mathbf{M} is simple transitive.

Remark 4.14. All results in this section can also be formulated and proven in the graded setup as, for example, considered in [MM3, Section 7] or [MT, Section 3].

5. MORITA–TAKEUCHI THEORY FOR (CO)ALGEBRA 1-MORPHISMS

Morita theory [Mo] explains that, under certain conditions, an equivalence of categories $\text{mod}(A) \cong \text{mod}(B)$, where A and B are algebras, can be given in terms of tensoring with an A - B bimodule. Takeuchi [Ta] “dualized” Morita theory for coalgebras, comodules and bicomodules. In this section, we explain how Morita(–Takeuchi) theory extends to (co)algebra, (co)module and bi(co)module 1-morphisms in finitary and fiat 2-categories.

5.1. The bicomodule story. Let \mathcal{C} be a finitary 2-category and A, B two coalgebra 1-morphisms in $\underline{\mathcal{C}}$. Suppose that $M = {}_A M_B$ and $N = {}_B N_A$ are bicomodule 1-morphisms in $\underline{\mathcal{C}}$ over A and B , with the left and right coaction 2-morphisms on the indicated sides denoted by λ_M, λ_N and ρ_M, ρ_N , respectively. Recall [Ta, §0] that the *cotensor product* $M \square_B N$ is defined as the kernel of the 2-morphism

$$M \circ N \xrightarrow{\rho_M \circ \text{id}_N - \text{id}_M \circ \lambda_N} M \circ B \circ N.$$

This is an A - A bicomodule 1-morphism in $\underline{\mathcal{C}}$. Similarly, one defines the B - B bicomodule 1-morphism $N \square_A M$.

The bicomodule 1-morphisms M and N induce functors:

$$- \square_A M : \text{comod}_{\underline{\mathcal{C}}}(A) \rightarrow \text{comod}_{\underline{\mathcal{C}}}(B) \quad \text{and} \quad - \square_B N : \text{comod}_{\underline{\mathcal{C}}}(B) \rightarrow \text{comod}_{\underline{\mathcal{C}}}(A).$$

Given two A - B bicomodule 1-morphisms M_1 and M_2 , there is a bijection

$$(\alpha : M_1 \rightarrow M_2) \mapsto (- \square_A \alpha : - \square_A M_1 \rightarrow - \square_A M_2)$$

between bicomodule 2-morphisms and the associated natural transformations.

It is easy to see that the cotensor product is associative (up to a canonical isomorphism) and that λ_M and ρ_M define A - B bicomodule 2-isomorphisms

$$M \xrightarrow{\cong} A \square_A M \quad \text{and} \quad M \xrightarrow{\cong} M \square_B B,$$

whose inverses are given by the counit 2-morphisms.

We say that a bicomodule 1-morphism is *biinjective* if it is injective as a left comodule 1-morphism and as a right comodule 1-morphism (but not necessarily as a bicomodule 1-morphism).

Theorem 5.1. *As 2-representations of \mathcal{C} , $\text{comod}_{\mathcal{C}}(A)$ and $\text{comod}_{\mathcal{C}}(B)$ are equivalent if and only if there exist biinjective bicomodule 1-morphisms $M = {}_A M_B$ and $N = {}_B N_A$ in \mathcal{C} and bicomodule 2-isomorphisms*

$$f: A \xrightarrow{\cong} M \square_B N \quad \text{and} \quad g: B \xrightarrow{\cong} N \square_A M,$$

such that the diagrams

$$(7) \quad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \square_B B \\ \lambda_M \downarrow & & \downarrow \text{id}_M \square g \\ A \square_A M & \xrightarrow{f \square \text{id}_M} & M \square_B N \square_A M \end{array} \quad \begin{array}{ccc} M & \xrightarrow{\rho_N} & N \square_A A \\ \lambda_N \downarrow & & \downarrow \text{id}_N \square f \\ B \square_B N & \xrightarrow{g \square \text{id}_N} & N \square_A M \square_B N \end{array}$$

commute.

Proof. The “if” part is clear, since the equivalence $\text{comod}_{\mathcal{C}}(A) \cong \text{comod}_{\mathcal{C}}(B)$ is defined by $-\square_A M$ and its inverse $-\square_B N$.

Next, we prove the “only if” part. This follows as in [Ta, §2 and §3], with only minor changes in the details of the proofs. Let $F: \text{comod}_{\mathcal{C}}(A) \rightarrow \text{comod}_{\mathcal{C}}(B)$ be an exact functor which intertwines the two 2-actions of \mathcal{C} . Then $F(A)$ is an A - B bicomodule 1-morphism and $F(-) \cong -\square_A F(A)$, following [Ta, Proposition 2.1]. Note that the proof is identical, except that we replace the preservation of direct sums by the intertwining property in the first sentence of Takeuchi’s proof. Similarly, if G is the inverse of F , then $G(B)$ is an B - A bicomodule 1-morphism and $G(-) \cong -\square_B G(B)$.

The natural isomorphisms

$$\text{id}_{\text{comod}_{\mathcal{C}}(B)} \xrightarrow{\cong} FG \quad \text{and} \quad \text{id}_{\text{comod}_{\mathcal{C}}(A)} \xrightarrow{\cong} GF$$

give rise to 2-isomorphisms $f: A \rightarrow F(A) \square_B G(B)$ and $g: B \rightarrow G(B) \square_A F(A)$ such that the squares in (7) commute. The same natural isomorphisms imply that F and G are biadjoint, so both functors are exact and send injectives to injectives. Therefore, $F(A)$ and $G(B)$ are biinjective (cf. [Ta, Theorem 2.5]). \square

The following corollary follows directly from Theorems 4.7 and 5.1.

Corollary 5.2. *Let \mathcal{C} be a finitary 2-category and \mathbf{M} a transitive 2-representation of \mathcal{C} . Furthermore, let $N_1, N_2 \in \mathbf{M}(\mathbf{i})$ be two objects as in Theorem 4.7. Then $\underline{\text{Hom}}(N_1, N_2)$ is a biinjective $\underline{\text{Hom}}(N_2, N_2)$ - $\underline{\text{Hom}}(N_1, N_1)$ bicomodule 1-morphism in \mathcal{C}*

and $\underline{\text{Hom}}(N_2, N_1)$ is a biinjective $\underline{\text{Hom}}(N_1, N_1)$ - $\underline{\text{Hom}}(N_2, N_2)$ bicomodule 1-morphism in $\underline{\mathcal{C}}$, such that

$$\begin{aligned}\underline{\text{Hom}}(N_1, N_1) &\cong \underline{\text{Hom}}(N_2, N_1) \square_{\underline{\text{Hom}}(N_2, N_2)} \underline{\text{Hom}}(N_1, N_2), \\ \underline{\text{Hom}}(N_2, N_2) &\cong \underline{\text{Hom}}(N_1, N_2) \square_{\underline{\text{Hom}}(N_1, N_1)} \underline{\text{Hom}}(N_2, N_1).\end{aligned}$$

5.2. The bimodule story. As already noted in Remark 3.1, if \mathcal{C} is fiat, the involution $*$ on \mathcal{C} gives rise to an equivalence between \mathcal{C} and $\overline{\mathcal{C}}$, which sends injective 1-morphisms to projective 1-morphisms. Using this fact, we see that Theorem 5.1 implies a “dual theorem” for algebra 1-morphisms in $\overline{\mathcal{C}}$, which we state below without further proof.

For this purpose, let M be an A - B bimodule 1-morphism in $\overline{\mathcal{C}}$, with left action 2-morphism $\lambda_M: A \circ M \rightarrow M$ and right action 2-morphism $\rho_M: M \circ B \rightarrow M$. Similarly, let N be a B - A bimodule 1-morphism. As before, define $M \circ_B N$ to be the cokernel of the 2-morphism

$$M \circ B \circ N \xrightarrow{\rho_M \circ \text{id}_N - \text{id}_M \circ \lambda_N} M \circ N.$$

Define $N \circ_A M$ similarly. Note that λ_M and ρ_M descend to isomorphisms

$$A \circ_A M \xrightarrow{\cong} M \quad \text{and} \quad M \circ_B B \xrightarrow{\cong} M,$$

respectively. The same holds for λ_N and ρ_N , of course.

We say that a bimodule 1-morphism is *biprojective* if it is projective as a left module 1-morphism and as a right module 1-morphism (but not necessarily as a bimodule 1-morphism).

Theorem 5.3. *Let \mathcal{C} be a fiat 2-category and A and B two algebra 1-morphisms in $\overline{\mathcal{C}}$. As 2-representations of $\overline{\mathcal{C}}$, $\text{mod}_{\overline{\mathcal{C}}}(A)$ and $\text{mod}_{\overline{\mathcal{C}}}(B)$ are equivalent if and only if there exist biprojective bimodule 1-morphisms $M = {}_A M_B$ and $N = {}_B N_A$ in $\overline{\mathcal{C}}$ and bimodule 2-isomorphisms*

$$f: M \circ_B N \xrightarrow{\cong} A \quad \text{and} \quad g: N \circ_A M \xrightarrow{\cong} B,$$

such that the diagrams

$$\begin{array}{ccc} M \circ_B N \circ_A M & \xrightarrow{f \circ \text{id}_M} & A \circ_A M \\ \text{id}_M \circ g \downarrow & & \downarrow \lambda_M \\ M \circ_B B & \xrightarrow{\rho_M} & M \end{array} \quad \begin{array}{ccc} N \circ_A M \circ_B N & \xrightarrow{g \circ \text{id}_N} & B \circ_B N \\ \text{id}_N \circ f \downarrow & & \downarrow \lambda_N \\ N \circ_A A & \xrightarrow{\rho_N} & N \end{array}$$

commute. □

Remark 5.4. Again, all results above admit generalizations to the graded setup.

An example of Morita(–Takeuchi) equivalent (co)algebra 1-morphisms is given later on in Example 6.3.

6. CONSTRUCTING (CO)ALGEBRA 1-MORPHISMS USING IDEMPOTENTS

6.1. Adjunctions and (co)monads. Recall the following (see, e.g. [ML]):

Let (F, G) be a pair of adjoint functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$, for two categories \mathcal{C} and \mathcal{D} . Let $\eta: 1_{\mathcal{C}} \rightarrow GF$ and $\epsilon: FG \rightarrow 1_{\mathcal{D}}$ be the corresponding unit and counit of the adjunction. Then the composition GF carries the natural structure of a monad given by η and $\text{id}_G \circ_0 \epsilon \circ_0 \text{id}_F: GF \rightarrow GF$. Further, the composition FG carries the natural structure of a comonad given by ϵ and $\text{id}_F \circ_0 \eta \circ_0 \text{id}_G: FG \rightarrow FG$. In particular, GF has the natural structure of an algebra 1-morphism in the 2-category of endofunctors of \mathcal{C} , and FG has the natural structure of a coalgebra 1-morphism in the 2-category of endofunctors of \mathcal{D} .

6.2. (Co)algebra 1-morphisms for projective bimodules. Let A be a finite dimensional algebra and $A\text{-mod-}A$ the bicategory of all A - A -bimodules (with respect to tensoring over A). Abusing notation, we will also denote by $A\text{-mod-}A$ its strictification. We also use the notation $A\text{-mod}$ and $\text{mod-}A$ in the evident way.

For an idempotent $e \in A$, consider the A - \mathbb{k} -bimodule Ae which defines an exact functor

$$F = Ae \otimes_{\mathbb{k}} -: \mathbb{k}\text{-mod} \rightarrow A\text{-mod}.$$

By the usual tensor-hom adjunction, the right adjoint of F is the functor

$$G = \text{Hom}_A(Ae, -): A\text{-mod} \rightarrow \mathbb{k}\text{-mod}.$$

The functor G is exact and is isomorphic to the functor

$$G' = eA \otimes_A -: A\text{-mod} \rightarrow \mathbb{k}\text{-mod}.$$

The morphisms corresponding to the counit and the unit of the pair (F, G) are

$$\varepsilon: Ae \otimes_{\mathbb{k}} eA \rightarrow A$$

given by multiplication and

$$\eta: \mathbb{k} \rightarrow eA \otimes_A Ae, \quad 1 \mapsto e \otimes e.$$

The exact functor F has a “left exact” representation via an isomorphic functor

$$F' = \text{Hom}_{\mathbb{k}}(\text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}), -): \mathbb{k}\text{-mod} \rightarrow A\text{-mod}.$$

Therefore, F has a left adjoint given by the functor

$$H = \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}) \otimes_A -: A\text{-mod} \rightarrow \mathbb{k}\text{-mod}.$$

The morphisms corresponding to the counit and the unit of the pair (H, F) are the evaluation morphism

$$\varepsilon': \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}) \otimes_A Ae \rightarrow \mathbb{k}, \quad \beta \otimes a \mapsto \beta(a)$$

and the coevaluation morphism

$$\eta': A \rightarrow Ae \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}), \quad 1 \mapsto \sum_i a_i \otimes a_i^*,$$

where $\{a_i\}$ is some fixed basis of Ae and $\{a_i^*\}$ is the corresponding dual basis of the \mathbb{k} -vector space $\text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$.

To state the next proposition, recall that an object A which is both an algebra and a coalgebra object, is called *Frobenius* provided that comultiplication is a homomorphism of A - A -bimodules, see e.g. [EGNO, Definition 7.20.3] and also [Mu, SF]. Henceforth, we use the notion of a *Frobenius 1-morphism*.

Proposition 6.1. *In the above setup we have:*

- (i) *The A - A -bimodule $Ae \otimes_{\mathbb{k}} eA$ has the structure of a coalgebra 1-morphism.*
- (ii) *If $eA \cong \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$ in $\text{mod-}A$, then the A - A -bimodule $Ae \otimes_{\mathbb{k}} eA$ has the structure of an algebra 1-morphism.*
- (iii) *In the setup of (ii), the A - A -bimodule $Ae \otimes_{\mathbb{k}} eA$ has the structure of a Frobenius 1-morphism.*

Proof. Claim (i) follows from the comonad discussion in Section 6.1 applied to the pair (F, G) of adjoint functors. Claim (ii) follows from the monad discussion in Section 6.1 applied to the pair (H, F) of adjoint functors.

To prove claim (iii), we have to check the Frobenius condition, that is compatibility of the algebra and coalgebra structures. This reduces to two commutative diagrams. For simplicity, we write $(Ae)^*$ for $\text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$. Then the first diagram is

$$\begin{array}{ccc} (Ae)^* \otimes_A Ae & \xrightarrow{\eta \otimes \text{id}_{(Ae)^*} \otimes_A Ae} & (Ae)^* \otimes_A Ae \otimes_{\mathbb{k}} (Ae)^* \otimes_A Ae \\ \varepsilon' \downarrow & & \downarrow \text{id}_{(Ae)^*} \otimes_A Ae \otimes \varepsilon' \\ \mathbb{k} & \xrightarrow{\eta} & (Ae)^* \otimes_A Ae \end{array}$$

and its commutativity is checked, using definitions, by the computation

$$\begin{array}{ccc} \beta \otimes a & \xrightarrow{\quad} & \Phi(e) \otimes e \otimes \beta \otimes a \\ \downarrow & & \downarrow \\ \beta(a) & \xrightarrow{\quad} & \beta(a) \Phi(e) \otimes e, \end{array}$$

where $\Phi: eA \xrightarrow{\cong} (Ae)^*$ is a fixed isomorphism.

The second diagram is

$$\begin{array}{ccc} (Ae)^* \otimes_A Ae & \xrightarrow{\text{id}_{(Ae)^*} \otimes_A Ae \otimes \eta} & (Ae)^* \otimes_A Ae \otimes_{\mathbb{k}} (Ae)^* \otimes_A Ae \\ \varepsilon' \downarrow & & \downarrow \varepsilon' \otimes \text{id}_{(Ae)^*} \otimes_A Ae \\ \mathbb{k} & \xrightarrow{\eta} & (Ae)^* \otimes_A Ae \end{array}$$

and its commutativity is checked, using definitions, by the computation

$$\begin{array}{ccc} \beta \otimes a & \xrightarrow{\quad} & \beta \otimes a \otimes \Phi(e) \otimes e \\ \downarrow & & \downarrow \\ \beta(a) & \xrightarrow{\quad} & \beta(a) \Phi(e) \otimes e. \end{array}$$

This completes the proof of the proposition. \square

6.3. Duflo involutions as (co)algebra 1-morphism. Let \mathcal{C} be a fiat 2-category and \mathcal{J} a two-sided cell in \mathcal{C} . Assume that \mathcal{C} is \mathcal{J} -simple and that \mathcal{J} is strongly regular. Let \mathcal{L} be a left cell in \mathcal{J} and G be the *Duflo involution* in \mathcal{L} . (For these notions, see [MM1, Proposition 17], [MM6, Propositions 27 and 28] and Section 2.8.)

Theorem 6.2. *In the above setting, we have:*

- (i) *The Duflo involution G is an algebra 1-morphism in \mathcal{C} . The corresponding 2-representation $\text{proj}_{\overline{\mathcal{C}}}(G)$ of \mathcal{C} is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$.*
- (ii) *The Duflo involution G is a coalgebra 1-morphism in \mathcal{C} . The corresponding 2-representation $\text{inj}_{\mathcal{C}}(G)$ of \mathcal{C} is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$.*
- (iii) *The Duflo involution G is a Frobenius 1-morphism in \mathcal{C} .*

Proof. We prove claim (i) and claim (ii) will follow by duality.

Since the only relevant 1-morphisms are those contained in \mathcal{J} , we may assume without loss of generality that \mathcal{J} contains all indecomposable 1-morphisms of \mathcal{C} that are not isomorphic to some identity 1-morphisms. Therefore, the classification of \mathcal{J} -simple 2-categories in [MM3, Theorem 13] reduces our statement to the special case $\mathcal{C} = \mathcal{C}_A$, for some weakly-symmetric finite dimensional basic algebra A . (\mathcal{C}_A is the 2-category whose objects are $A_i\text{-mod}$, with A_i being the connected components of A , and whose morphism categories are generated by functors isomorphic to tensoring with projective $A_i\text{-}A_j$ -bimodules. See e.g. [MM1, Section 7.3] for details on \mathcal{C}_A .) In the case $\mathcal{C} = \mathcal{C}_A$, the Duflo involution G has the form $Ae \otimes_{\mathbb{k}} eA$, for some primitive idempotent $e \in A$, see e.g. [MM6, Proposition 28]. Therefore, existence of the algebra 1-morphism structure on G follows immediately from Proposition 6.1(ii).

It remains to prove the claim about 2-representations. Let $e = e_1, e_2, \dots, e_n$ be a complete list of pairwise orthogonal primitive idempotents of A . Then

$$\{Ae_i \otimes eA : i = 1, 2, \dots, n\}$$

is the list of all 1-morphisms in \mathcal{L} , up to isomorphism. By [MM1, Section 7.3], the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$ is equivalent to the defining action of \mathcal{C}_A on the category $A\text{-proj}$ of projective A modules.

Since A is a self-injective and weakly symmetric algebra, there is an isomorphism $eA \cong \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$ of right A -modules which we may use to identify eA and $\text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$. Hence, we may express the unit morphism

$$\eta: A \rightarrow Ae \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$$

in the form $\eta(1) = \sum_i a_i \otimes a_i^*$, where $\{a_i\}$ is some basis of Ae and $\{a_i^*\}$ is the corresponding dual basis of $\text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$. Furthermore, the multiplication map

$$(Ae \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})) \otimes_A (Ae \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})) \rightarrow Ae \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$$

is just given by contraction, i.e. $(a \otimes \varphi) \otimes (b \otimes \psi) \mapsto \varphi(b)(a \otimes \psi)$.

Each $Ae_i \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$, $i = 1, 2, \dots, n$, is naturally a right G -module, and the additive closure of these objects is clearly stable under the left \mathcal{C}_A -action. Let us denote the resulting 2-representation of \mathcal{C}_A by \mathbf{M} . Further, the fact that homomorphisms in $\text{mod}_{\mathcal{C}_A}(G)$ must commute with the G -action implies that

$$\text{Hom}_{\text{mod}_{\mathcal{C}_A}(G)}(Ae_i \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}), Ae_j \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(Ae, \mathbb{k}))$$

is the $e_i A e_j \otimes_{\mathbb{k}} e$ subalgebra of

$$\mathrm{Hom}_{\mathcal{C}_A}(Ae_i \otimes_{\mathbb{k}} \mathrm{Hom}_{\mathbb{k}}(Ae, \mathbb{k}), Ae_j \otimes_{\mathbb{k}} \mathrm{Hom}_{\mathbb{k}}(Ae, \mathbb{k})) \cong e_i A e_j \otimes_{\mathbb{k}} e A e.$$

This means that the Cartan matrix of the underlying algebra of \mathbf{M} coincides with the Cartan matrix of A . Thus, [MM6, Theorem 4] implies that \mathbf{M} is equivalent to $\mathbf{C}_{\mathcal{L}}$.

On the other hand, any $X \in \mathrm{mod}_{\mathcal{C}_A}(\mathbf{G})$ is a quotient of $X A e \otimes_{\mathbb{k}} \mathrm{Hom}_{\mathbb{k}}(Ae, \mathbb{k})$, by the argument dual to the one used in the last paragraph of the proof of Theorem 4.7. This implies that \mathbf{M} coincides with the 2-representation $\mathrm{proj}_{\overline{\mathcal{C}}}(\mathbf{G})$ of \mathcal{C} and completes the proof of claim (i).

Claim (iii) follows from the above and Proposition 6.1(iii). \square

Example 6.3. For the class of (co)algebra 1-morphisms in Proposition 6.1, it is not hard to work out the Morita(-Takeuchi) 2-theory from Section 5 explicitly.

Suppose that e and f are two non-zero primitive idempotents in a finite dimensional self-injective algebra A . It follows from the proof of Theorem 6.2 that the corresponding coalgebra 1-morphisms

$$E_e = Ae \otimes_{\mathbb{k}} eA \quad \text{and} \quad E_f = Af \otimes_{\mathbb{k}} fA$$

are Morita–Takeuchi equivalent. More explicitly, the equivalence and its inverse are given by

$$\begin{aligned} - \square_{E_e}(Ae \otimes_{\mathbb{k}} fA) : \mathrm{comod}_{A\text{-mod-}A}(E_e) &\xrightarrow{\cong} \mathrm{comod}_{A\text{-mod-}A}(E_f) \\ - \square_{E_f}(Af \otimes_{\mathbb{k}} eA) : \mathrm{comod}_{A\text{-mod-}A}(E_f) &\xrightarrow{\cong} \mathrm{comod}_{A\text{-mod-}A}(E_e). \end{aligned}$$

Moreover, E_e and E_f are also Morita equivalent algebra 1-morphisms. This time, the equivalence and its inverse are given by

$$\begin{aligned} - \circ_{E_e}(Ae \otimes_{\mathbb{k}} fA) : \mathrm{mod}_{A\text{-mod-}A}(E_e) &\xrightarrow{\cong} \mathrm{mod}_{A\text{-mod-}A}(E_f) \\ - \circ_{E_f}(Af \otimes_{\mathbb{k}} eA) : \mathrm{mod}_{A\text{-mod-}A}(E_f) &\xrightarrow{\cong} \mathrm{mod}_{A\text{-mod-}A}(E_e). \end{aligned}$$

6.4. Wall-crossings as Frobenius 1-morphism. The constructions described in the previous subsections admit a generalization as follows.

Let A be a finite dimensional unital \mathbb{k} -algebra and B be a unital subalgebra of A . Assume that A is projective, both as a left and as a right B -module, and that both algebras, A and B , are symmetric in the sense that there exist bimodule isomorphisms ${}_A A_A \cong {}_A \mathrm{Hom}_{\mathbb{k}}(A, \mathbb{k})_A$ and ${}_B B_B \cong {}_B \mathrm{Hom}_{\mathbb{k}}(B, \mathbb{k})_B$. Then the induction and restriction functors

$${}_A A \otimes_B - : B\text{-mod} \rightarrow A\text{-mod} \quad \text{and} \quad {}_B A \otimes_A - : A\text{-mod} \rightarrow B\text{-mod}$$

are biadjoint, cf. [MM6, Section 6.4]. Consequently, similarly to Proposition 6.1, the A - A -bimodule ${}_A A \otimes_B {}_A A$ has the structure of a Frobenius 1-morphism.

The above can be applied to the following situation: Let (W, S) be a finite Coxeter system with a fixed reflection representation \mathfrak{h} of W and \mathbf{C} the corresponding coinvariant algebra. Let $\mathcal{S} = \mathcal{S}(W, S, \mathfrak{h})$ denote the associated 2-category of *Soergel \mathbf{C} - \mathbf{C} -bimodules*, see [So1, So2, EW]. Then the indecomposable Soergel bimodules are naturally indexed by elements in W and, for $w \in W$, we denote by B_w the corresponding indecomposable Soergel bimodule.

For $X \subset S$, let W_X be the corresponding parabolic subgroup of W . Denote by w_0^X the longest element in W_X .

Proposition 6.4. *The \mathbb{C} - \mathbb{C} -bimodule $B_{w_0^X}$ has the natural structure of a Frobenius 1-morphism in \mathcal{S} .*

Proof. Let \mathbb{C}^X denote the algebra of X -invariants in \mathbb{C} . Then both, \mathbb{C} and \mathbb{C}^X , are symmetric algebras and \mathbb{C} is projective as a left and as a right \mathbb{C}^X -module, see e.g. [Hi]. Moreover, the \mathbb{C} - \mathbb{C} -bimodule $B_{w_0^X}$ is isomorphic to $\mathbb{C} \otimes_{\mathbb{C}^X} \mathbb{C}$, see e.g. [So1, Section 3.4] which also admits a straightforward generalization to finite Coxeter groups. The claim follows from the discussion in Section 6.2. \square

Remark 6.5. As in Remarks 4.14 and 5.4, the results above generalizes without difficulties to the graded world.

7. APPLICATION TO SOERGEL BIMODULES FOR DIHEDRAL GROUPS

In this section we work over $\mathbb{k} = \mathbb{C}$. Moreover, we fix a positive integer $n > 2$.

7.1. Various 2-categories: from affine \mathfrak{sl}_2 to singular Soergel bimodules. We denote by \mathcal{A}_n the category of representations of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ at level $n-2$, see [Ka] and also [Os, Section 6] and references therein for details. The category \mathcal{A}_n has the structure of a (non-strict) monoidal category via the so-called *fusion product*, see e.g. [Fi, Section 2.11]. We can therefore consider \mathcal{A}_n as a bicategory. Abusing notation, we will also denote by \mathcal{A}_n the strictification of the latter bicategory.

For a primitive complex $2n$ -th root of unity q , consider also the semisimple subquotient \mathcal{Q}_n of the category of integrable representations of the quantum group $U_q(\mathfrak{sl}_2)$, see e.g. [GK, An, AP]. The category \mathcal{Q}_n can be seen as the additive closure of simple finite dimensional (highest weight) modules L_k of quantum dimension $[k+1]_q$, where $k = 0, 1, \dots, n-2$. Again, \mathcal{Q}_n has the structure of a (non-strict) monoidal category given by the quantum *fusion product*. Abusing notation, we will also denote by \mathcal{Q}_n the strictification of the latter bicategory.

According to [Fi], the 2-categories \mathcal{Q}_n and \mathcal{A}_n are equivalent. This equivalence is based on the results of [KL1, KL2, KL3, KL4, Lu].

Moreover, \mathcal{Q}_n also has a diagrammatic presentation, because it is equivalent to the Karoubi envelope of the one-object 2-category of all Temperley-Lieb diagrams modulo the 2-ideal of the negligible ones (see e.g. [Tu, Section XII.7]). Elias [El, Section 4] uses a closely related 2-category, denoted $\widetilde{\mathcal{TL}}_n(\delta)$ and called the *two-color Temperley-Lieb 2-category*, with the two objects (“colors”) s and t . In \mathcal{TL}_n , the regions of the Temperley-Lieb diagrams are colored by s or t , such that any two regions separated by one strand have different colors. This ensures that the coloring of any diagram is uniquely determined by the color of its rightmost region. Vice versa, we can extend any color of the rightmost region of given diagram uniquely to a coloring of the whole diagram. Finally, let \mathcal{TL}_n denote the quotient of $\widetilde{\mathcal{TL}}_n(q + q^{-1})$ by the 2-ideal generated by the negligible Jones-Wenzl projector corresponding to our choice of n , see [El, Section 4] for details. (Here q is the same root of unity as before.)

Consider also the bicategory $\widetilde{\mathcal{SS}}_n$ of *singular Soergel bimodules* (over the polynomial algebra) for the dihedral group

$$D_{2n} = \langle s, t \mid s^2 = t^2 = 1, \underbrace{sts \cdots}_{n \text{ factors}} = \underbrace{tst \cdots}_{n \text{ factors}} \rangle$$

of order $2n$, and denote by \mathcal{SS}_n the quotient of $\widetilde{\mathcal{SS}}_n$ by the 2-ideal generated by all D_{2n} -invariant polynomials of positive degree. Then \mathcal{SS}_n is the bicategory of singular Soergel bimodules over the coinvariant algebra \mathbb{C} of D_{2n} . We refer to [El, Section 6.1] and [Wi] for details. (We also use a notation similar to that in Section 6.4.) We will again denote the corresponding strictifications by the same symbols.

We denote by $\widehat{\mathcal{SS}}_n$ the *small quotient* of \mathcal{SS}_n , that is the quotient of \mathcal{SS}_n by the 2-ideal generated by the indecomposable Soergel bimodule corresponding to the longest element in D_{2n} . Further, we denote by \mathcal{S}_n the endomorphism 2-category of the regular object in \mathcal{SS}_n , that is the 2-category of (regular or usual) Soergel bimodules and similarly for $\widehat{\mathcal{S}}_n$. Again, we denote their corresponding strictifications by the same symbols.

Remark 7.1. It follows from Corollary 4.10 that there is a bijection between simple transitive and faithful 2-representations of $\widehat{\mathcal{S}}_n$ and $\widehat{\mathcal{SS}}_n$. Note that Corollary 4.10 is applicable due to the fact that the identity 1-morphism on the s -singular object of $\widehat{\mathcal{SS}}_n$ factors through $\widehat{\mathcal{S}}_n$, which implies that every faithful 2-representation of $\widehat{\mathcal{SS}}_n$ must be supported, in particular, on this s -singular object. (Similarly for t .)

7.2. Based modules and algebra 1-morphisms for affine \mathfrak{sl}_2 . The papers [DZ] and [EK] study and classify so-called indecomposable *based* or \mathbb{Z}_+ *modules* over the split Grothendieck group $[\mathcal{A}_n]_{\oplus}$ of \mathcal{A}_n (also called the *Verlinde algebra*). Such modules turn out to be in one-to-one correspondence with finite Dynkin diagrams and so-called tadpole diagrams which have Coxeter number n .

Theorem 7.2. ([Os, Theorem 6.1], see also [BEK].) *For each simply laced Dynkin diagram Γ with Coxeter number n , there is a unique, up to isomorphism, algebra 1-morphism A_{Γ} in \mathcal{A}_n such that the based $[\mathcal{A}_n]_{\oplus}$ -module $[\mathrm{proj}_{\mathcal{A}_n}(A_{\Gamma})]_{\oplus}$ corresponds to Γ via the equivalence in [EK].*

We stress that for the non-simply laced finite Dynkin diagrams and the tadpole diagrams appearing in [EK, Section 3.3] the corresponding algebra 1-morphisms do not exist.

7.3. Algebra 1-morphisms for Soergel bimodules. Certain indecomposable based modules over the split Grothendieck group $[\mathcal{S}_n]_{\oplus}$ are in one-to-one correspondence with Dynkin diagrams of ADE type and Coxeter number n , as it was shown in [KMMZ, Sections 6 and 7]. There are corresponding algebra 1-morphisms:

Theorem 7.3. *For each simply laced Dynkin diagram Γ with Coxeter number n , there is an algebra 1-morphism B_{Γ} in $\widehat{\mathcal{S}}_n$ such that the based $[\widehat{\mathcal{S}}_n]_{\oplus}$ -module $[\mathrm{proj}_{\widehat{\mathcal{S}}_n}(B_{\Gamma})]_{\oplus}$ corresponds to Γ via the bijection in [KMMZ, Sections 6 and 7].*

Proof. First consider the algebra 1-morphism B_{Γ}^s given by the image of A_{Γ} from Theorem 7.2 via the composition of the following 2-functors:

- Finkelberg's equivalence between the two 2-categories \mathcal{A}_n and \mathcal{Q}_n ;
- the fully-faithful embedding of the 2-category \mathcal{Q}_n into $\mathcal{T}\mathcal{L}_n$ given by

$$\underbrace{L_1 \otimes \cdots \otimes L_1}_k \mapsto \underbrace{\dots ststs}_k, \quad k = 0, 1, \dots, n-2,$$

with the usual assignments on morphisms, where L_1 denotes the vector representation of $U_q(\mathfrak{sl}_2)$ (which monoidally generates \mathcal{Q}_n);

- the faithful 2-functor from the 2-category $\mathcal{T}\mathcal{L}_n$ to $\widehat{\mathcal{T}\mathcal{T}}_n$ defined by Elias in [El, Proposition 1.2 and Theorem 6.29], which is also full onto degree-zero 2-morphisms, composed with the projection onto $\widehat{\mathcal{T}\mathcal{T}}_n$.

Notice that the second functor is the composite of the equivalence between \mathcal{Q}_n and the quotient of the Temperley-Lieb 2-category, mentioned in Section 7.1, and the fully-faithful embedding of the latter into $\mathcal{T}\mathcal{L}_n$ defined by coloring the right-most region of every diagram by s . (Alternatively, one could color it by t , of course.)

Let us check that the indecomposable based $[\widehat{\mathcal{T}\mathcal{T}}_n]_{\oplus}$ -module $[\text{proj}_{\widehat{\mathcal{T}\mathcal{T}}_n}(B_{\Gamma}^s)]_{\oplus}$ has the correct combinatorics.

To this end, we recall that all involved 2-categories have a natural positive grading and all involved 2-functors are gradable. (Note that our whole setup is applicable so far, cf. Remarks 4.14, 5.4 and 6.5.) The correctness of the combinatorics in question follows if we can show that the (graded version of) the above composition is full on 2-morphisms of degree zero. Indeed, if the pushforward Y of an indecomposable projective X in $\text{proj}_{\mathcal{A}_n}(A_{\Gamma})$ were decomposable, that would mean existence of a non-trivial idempotent in the endomorphism ring of Y , considered as a 1-morphism in $\text{proj}_{\widehat{\mathcal{T}\mathcal{T}}_n}(B_{\Gamma}^s)$. Next, from fullness on degree zero 2-morphism, we would get a non-trivial idempotent in the endomorphism ring of X , considered as a 1-morphism of \mathcal{A}_n . However, due to faithfulness of the above composition, this idempotent will also live in $\text{proj}_{\mathcal{A}_n}(A_{\Gamma})$, a contradiction.

Hence, it remains to check fullness of the three 2-functors from above on 2-morphisms of degree zero. For the first 2-functor of the above composition the claim about fullness on 2-morphisms of degree zero is clear. For the last 2-functor of the composition this is contained in [El, Proposition 1.2]. For the second 2-functor, such a claim follows directly from the definitions.

By construction and the properties of Elias' 2-functor from [El, Proposition 1.2], B_{Γ}^s lives in the endomorphism 2-category of the s -singular object in $\widehat{\mathcal{T}\mathcal{T}}_n$. Let

$$B_{\Gamma} = \mathbb{C} \otimes_{\mathbb{C}^s} B_{\Gamma}^s \otimes_{\mathbb{C}^s} \mathbb{C},$$

where, as above, \mathbb{C} is the coinvariant algebra of D_{2n} and \mathbb{C}^s the subalgebra which is the quotient of the subalgebra of s -invariant polynomials. Note that

$$B_{\Gamma} \otimes_{\mathbb{C}} B_{\Gamma} \cong \mathbb{C} \otimes_{\mathbb{C}^s} B_{\Gamma}^s \otimes_{\mathbb{C}^s} \mathbb{C} \otimes_{\mathbb{C}^s} B_{\Gamma}^s \otimes_{\mathbb{C}^s} \mathbb{C}$$

maps to

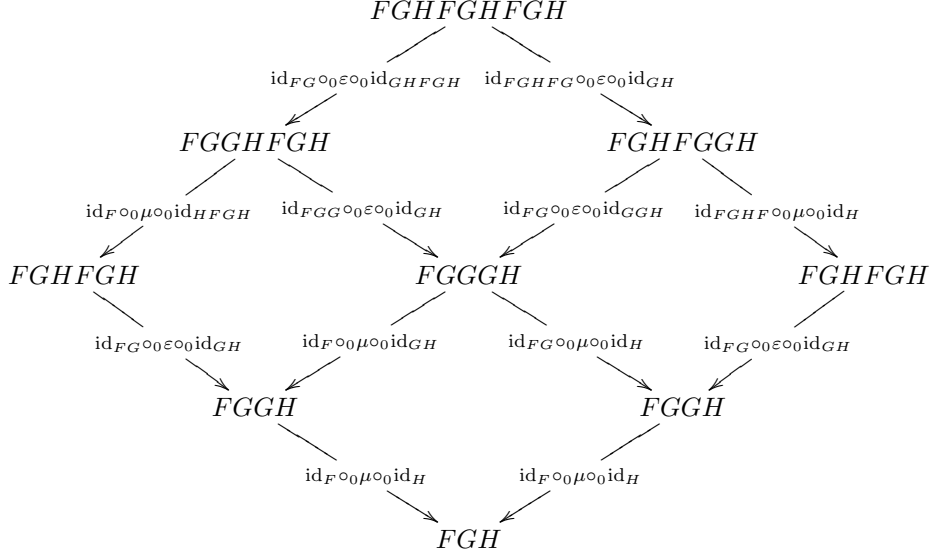
$$\mathbb{C} \otimes_{\mathbb{C}^s} B_{\Gamma}^s \otimes_{\mathbb{C}^s} B_{\Gamma}^s \otimes_{\mathbb{C}^s} \mathbb{C},$$

by applying the adjunction morphism given by the Demazure operator $\partial_s: \mathbb{C} \rightarrow \mathbb{C}^s$ (see [El, Section 3.6] for the definition) to the tensor factor in the middle (we are omitting

gradings, for simplicity). Composing this map with the multiplication morphism of B_Γ^s gives the multiplicative structure on B_Γ .

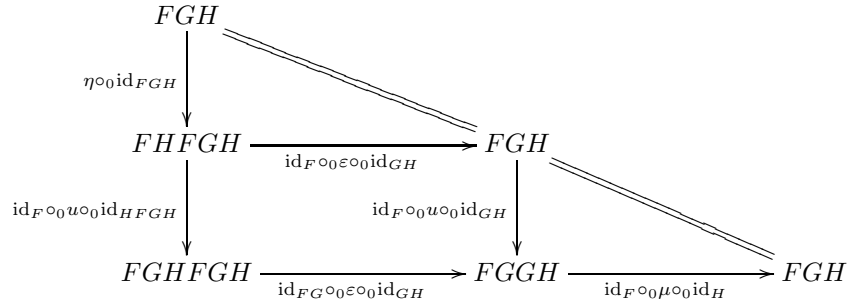
The unital structure $\mathbb{C} \rightarrow B_\Gamma$ is obtained from the unital structure $\mathbb{C}^s \rightarrow B_\Gamma^s$, by tensoring on both sides with \mathbb{C} over \mathbb{C}^s and precomposing with usual adjunction bimodule map $\mathbb{C} \rightarrow \mathbb{C} \otimes_{\mathbb{C}^s} \mathbb{C}$ given by $1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$, where α_s is the simple root corresponding to the reflection s (see e.g. [El, Example 3.4]).

To prove associativity of multiplication for B_Γ , consider the following diagram:



Here G stands for the algebra 1-morphism B_Γ^s and $\mu: GG \rightarrow G$ for the corresponding multiplication 2-morphism. Further, F and H denote translations out of the s -wall (i.e. ${}_c\mathbb{C}_{\mathbb{C}^s} \otimes_{\mathbb{C}^s} -$) and to the s -wall (i.e. ${}_c\mathbb{C}_{\mathbb{C}^s} \otimes_{\mathbb{C}^s} -$), respectively, with $\varepsilon: HF \rightarrow \text{Id}$ being the counit of the adjunction (given by ∂_s). The top rhombus and the two rhombi on the sides commute by the interchange law. The bottom rhombus commutes due to associativity of μ . Therefore the whole diagram commutes which yields associativity of multiplication for B_Γ .

To prove unitality of B_Γ consider the diagram



with the same notation as above and, additionally, where $u: \text{Id} \rightarrow G$ denotes the unit morphism for G and $\eta: \text{Id} \rightarrow FH$ is the unit of the adjunction. Here the top triangle commutes by adjunction, the square commutes by the interchange law and the right

triangle commutes due to unitality of B_Γ^s . Therefore the whole diagram commutes and proves unitality of B_Γ . Hence, B_Γ is indeed an algebra 1-morphism.

Since we have $\mathbb{C} \cong \mathbb{C}^s\{1\} \oplus \mathbb{C}^s\{-1\}$ as \mathbb{C}^s - \mathbb{C}^s -bimodules, the restriction of B_Γ to \mathbb{C}^s (on both sides) is isomorphic to $B_\Gamma^s\{2\} \oplus B_\Gamma^s \oplus B_\Gamma^s \oplus B_\Gamma^s\{-2\}$ as an algebra 1-morphism. This implies that the simple transitive 2-representation of $\widehat{\mathcal{S}}_n$ corresponding to the algebra 1-morphism B_Γ can be obtained from the one of $\widehat{\mathcal{S}\mathcal{S}}_n$ corresponding to B_Γ^s by restriction as in Corollary 4.10. \square

Thanks to Theorem 4.7 and the construction in [MM6, Section 3.6], there are always many non-equivalent 2-representations of $\widehat{\mathcal{S}}_n$ whose Grothendieck groups give rise to the same based $[\widehat{\mathcal{S}}_n]_\oplus$ -module. (These are obtained by an inflation process from the ones constructed via Theorem 7.3.) Therefore one cannot expect any direct uniqueness statement in Theorem 7.3 similar to the one of Theorem 7.2. However, we expect that under some additional assumptions of simplicity together with specification of $\mathfrak{t} \in \mathcal{C}$ identifying the endomorphism 2-category in which the algebra 1-morphism lives, the algebra 1-morphism in the formulation of Theorem 7.3 should be unique (in which case it, most probably, will not be the one constructed in the proof but rather a “simple quotient” of the latter). See also [MT, Theorem II].

7.4. Some concluding remarks.

Remark 7.4. If Γ is of Dynkin type A , then the corresponding simple transitive 2-representation of $\widehat{\mathcal{S}}_n$ is equivalent to a cell 2-representation (in the sense of Section 2.8), as was shown in [KMMZ, Sections 6 and 7]. Furthermore, by the results in [KO, Table 1] and Theorem 7.3, we have $B_\Gamma = B_s$. The algebra structure on B_s is given by the usual degree zero bimodule map $B_s \otimes_{\mathbb{C}} B_s \rightarrow B_s\{1\}$ of the Soergel calculus (the one which corresponds to the “merge” in Elias’ two-color Soergel calculus, cf. [El, Section 5.3]).

If Γ is of Dynkin type D , then the simple transitive 2-representation of $\widehat{\mathcal{S}}_n$ can be constructed from a cell 2-representation using the orbits under an involution. This was shown in [KMMZ, Section 7]. By the results in [KO, Table 1] and Theorem 7.3, we have $B_\Gamma = B_s \oplus B_{sw_0}$. (Here w_0 denotes the longest word in the dihedral group D_{2n} .) The 2-morphisms which define the algebra structure on B_Γ , can be deduced from the results in [KO, Section 7] and the relation between Temperley-Lieb diagrams and Elias’ two-color Soergel calculus. It is not hard to check by hand that these induce an algebra structure on B_Γ . (See also Remark 7.5.)

If Γ is of Dynkin type E , the situation is different. The existence of the corresponding simple transitive 2-representation of $\widehat{\mathcal{S}}_n$ was predicted in [KMMZ, Section 7.5] and its construction was given in [MT] using Elias’ diagrammatic two-color Soergel calculus. From the results in [KO, Table 1] and Theorem 7.3, we obtain the decomposition of B_Γ into indecomposable Soergel bimodules. However, the 2-morphisms which define the algebra structure on B_Γ are only determined up to scalars. Fixing these scalars is hard. In [KO, Theorem 6.1] and [Os, Theorem 6.1] it is shown that this is possible in a roundabout way, using arguments from conformal field theory. We were unable to prove the analogous result in $\widehat{\mathcal{S}}_n$ directly and by hand due to the complexity of the diagrams involved. It would be interesting to have such a proof.

Remark 7.5. We stress another conceptual difference between the B_Γ for Γ of Dynkin types A and D on one side, and those for Γ of Dynkin type E on the other: Using the description of the multiplication of Kazhdan–Lusztig basis elements in $[\widehat{\mathcal{S}}_n]_\oplus$ (see e.g. [dC, Section 4]), it is not hard to see that the B_Γ for types A and D descend to (pseudo-)idempotents in $[\widehat{\mathcal{S}}_n]_\oplus$. However, this is not true for the B_Γ of type E .

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